8.2 Dedekind domains

Def. A Dedekind domain is a domain A that is noetherian, integrally closed, and $\dim A \leq 1$.

⚠ Some authors require $\dim A = 1$, this only excludes fields.

Exm: $\mathbb{Z}$, every PID, $\mathbb{R}[x,y]/(x^2+y^2-1)$

Prop 8.4 Let A be a domain. TFAE

(a) A is a Dedekind domain.

(b) A is noetherian and $\forall P \in \operatorname{Spec}(A)\setminus\{0\}$: A_P is a DVR

(c) A is noetherian, ≤ 1-dimensional, and $\forall P \in \operatorname{Spec}(A)$: A_P regular

Proof: (a)$\Rightarrow$(b) A_P is i.c., noeth., local, 1-dim, so a DVR (T8.3)

(b)$\Rightarrow$(c) $A_0 = \mathcal{F}(A)$ is regular; each A_P, $P \neq 0$, is regular by T8.3

$\dim A = \sup\{\dim A_P : P \in \operatorname{Spec}(A)\} \leq 1$.

(c)$\Rightarrow$(a) A is i.c., because A_P is i.c. $\forall P \in \operatorname{Spec}(A)$ (T8.3, C6.6). $\square$

Remark (1) K alg. closed field, $X \subseteq \mathbb{A}^n(K)$ a variety. Then

$A(X)$ is a Dedekind domain $\Leftrightarrow$ X is an irreducible curve [$=\dim 1$] that is regular at every point.

(2) If $K/\mathbb{Q}$ is a finite field extension (= a number field), then

$\mathcal{O}_K = cl_K(\mathbb{Z})$ is i.c., 1-dim., noetherian [Exc Set 10, WS on Number Theory] $\Rightarrow$ $\mathcal{O}_K$ is a Dedekind domain.

$\mathcal{O}_K$ .... ring of algebraic integers in K

Prop 8.5 A Dedekind domain, not a field.

(1) If $P \in \operatorname{Max}(A)$ and $Q \trianglelefteq A$, then

Q is P-primary $\Leftrightarrow$ $\exists n \geq 1$: $Q = P^n$

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(2) Every $0 \neq I \trianglelefteq A$ has a prime factorization

$$I = P_1^{k_1} \cdot \ldots \cdot P_r^{k_r} \quad \text{with pw. distinct } P_i \in \mathrm{Max}(A),\ r \geq 0,\ k_i > 0.$$

This repr is unique (up to permuting factors), $P_1, \ldots, P_r$ are the associated primes of I.

<u>Proof</u>: (1) „$\Leftarrow$" P^n is P-primary by maximality of P [L5.1].

„$\Rightarrow$" Consider the localization $j: A \to A_P$. (A_P is a DVR)

$$\Rightarrow 0 \neq QA_P \underset{\uparrow \text{because } Q \subseteq P}{\ntrianglelefteq} A_P \overset{P8.1}{\Longrightarrow} QA_P = (PA_P)^n = P^n A_P \text{ for some } n \geq 1.$$

$$\Rightarrow Q \overset{L5.8}{=} j^{-1}(QA_P) = j^{-1}(P^n A_P) \overset{L5.8}{=} P^n$$

(2) Existence: A noetherian, 1-dim $\overset{C5.11}{\Longrightarrow}$ $I = Q_1 \cdots Q_r$ with primary ideals Q_i, and $\underset{= Q_1 \cap \ldots \cap Q_r}{}$

$\sqrt{Q_i} \neq \sqrt{Q_j}$ for $i \neq j$.

$P_i := \sqrt{Q_i} \overset{(1)}{\Longrightarrow} Q_i = P_i^{k_i}$ for some $k_i \geq 1$

$\Rightarrow I = P_1^{k_1} \cdots P_r^{k_r}$

<u>Uniqueness</u>: Suppose also $I = (P_1')^{e_1} \cdots (P_s')^{e_s}$, $P_i' \in \mathrm{Max}(A)$ pw. distinct, $e_i \geq 1$. $\overset{CRT}{\Longrightarrow} I = (P_1')^{e_1} \cap \ldots \cap (P_s')^{e_s} \overset{T5.9}{\Longrightarrow} r = s$ and, after renumbering, $P_i^{k_i} = (P_i')^{e_i}$, and $P_i = P_i'$ (take radicals), so $P_i^{k_i} = P_i^{e_i}$.

In the DVR A_{P_i}: $\quad (P_i A_{P_i})^{k_i} = (P_i A_{P_i})^{e_i} \overset{P8.1}{\Longrightarrow} k_i = e_i$. $\square$

<u>Def</u>: A domain, $K = \mathcal{F}(A)$. A fractional ideal is an A-submodule I of K s.t. there exists $0 \neq d \in A$: $\quad dI \subseteq A$.

<u>Exm</u>: (1) If $I \subseteq A$, then I is a frac. ideal $\Leftrightarrow$ I is an ideal ($\Leftarrow$: take $d = 1$)

(2) If $x_1, \ldots, x_n \in K$, then $\langle x_1, \ldots, x_n \rangle_A \leq K$ is a fractional ideal (take a a common denominator of the x_i). E.g. $\frac{1}{2}\mathbb{Z}$.

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Remark: (1) I is a frac. ideal $\Leftrightarrow$ $I = d^{-1}I'$ for some $d \in A\setminus\{0\}$, $I' \trianglelefteq A$.

(2) Suppose A is noetherian. Then: I is a frac. ideal

$\Leftrightarrow \exists x_1, \dots, x_n \in K$: $I = \langle x_1, \dots, x_n \rangle_A$

[„$\Leftarrow$" above „$\Rightarrow$" $dI \trianglelefteq A \Rightarrow dI = (y_1, \dots, y_n) \Rightarrow I = \langle d^{-1}y_1, \dots, d^{-1}y_n \rangle_A$.]

Def: A domain, $K = \mathfrak{F}(A)$. Let $I, J \leq K_A$ ← A-submodules of K

$IJ := {}_A\langle \{xy : x \in I, y \in J\} \rangle$, $(I :_K J) := \{x \in K : xJ \subseteq I\}$

(1) I is an invertible frac. ideal if $\exists I' \leq K_A$: $II' = A$

(2) I is principal if $I = Ax$ for some $x \in K$.

Lemma 8.6 (1) If I, J are frac. ideals, so is IJ. If additionally $J \neq \underline{0}$, then $(I :_K J)$ is a frac. ideal.

(2) Let $I \leq K_A$. If $J \leq K_A$ s.t. $IJ = A$, then $J = (A :_K I)$.

(3) I non-zero principal $\overset{i)}{\Rightarrow}$ I invertible frac. ideal $\overset{ii)}{\Rightarrow}$ I frac. ideal & finitely generated/A

Proof: (1) $IJ \leq K_A$, $(I :_K J) \leq K_A$ are easily checked.

Suppose $aI \subseteq A$, $bJ \subseteq A$ for $a, b \in A\setminus\{0\}$. $\Rightarrow$ $abIJ \subseteq A$, so IJ is a frac. ideal.

Suppose $J \neq \underline{0}$, let $0 \neq c \in J$.

$$\Rightarrow (I :_K J)\overset{\neq 0}{\overbrace{ca}} \subseteq Ia \subseteq A.$$

(2) $J \subseteq (A :_K I)$, so $A = IJ \subseteq I(A :_K I) \subseteq A$

$$\Rightarrow IJ = I(A :_K I) \overset{\cdot J}{\Rightarrow} A \cdot J = A \cdot (A :_K I) \Rightarrow J = (A :_K I).$$

(3) i) Let $I = xA$, $x \neq 0$ $\Rightarrow$ $J = x^{-1}A$ satisfies $IJ = A$

ii) Suppose I is invertible $\overset{(2)}{\Rightarrow}$ $I(A :_K I) = A$

$$\Rightarrow \exists n \geq 1, a_i \in I, b_i \in (A :_K I) : \quad 1 = \sum_{i=1}^{n} a_i b_i$$

$$\Rightarrow \forall b \in I: b = 1 \cdot b = \sum_{i=1}^{n} a_i \underbrace{(b_i b)}_{\in A} \Rightarrow I = \langle a_1, \dots, a_n \rangle_A \text{ is f.g.}$$

$\Rightarrow \forall b \in I: b = 1 \cdot b = \sum_{i=1} a_i \underbrace{(b_i b)}_{\in A} \Rightarrow I = \langle a_1, \dots, a_n \rangle_A$ is f.g. as A-module. If $0 \neq d \in A$ is a common denominator of $a_1, \dots, a_n$, then $dI \subseteq A$. $\square$

Def: For $I \leq K_A$, let $I^{-1} := (A :_K I)$ (Note: If $I \subseteq A \Rightarrow A \subseteq I^{-1}$)

Lemma 8.7 A domain.

Suppose $P_1, \dots, P_r, Q_1, \dots, Q_s \in \mathrm{Spec}(A)$ are invertible and $P_1 \cdots P_r = Q_1 \cdots Q_s$. Then $r = s$ and, up to permutation, $P_i = Q_i$ $\forall 1 \leq i \leq r$.

Proof: Induction on r. $r=0$: $\Rightarrow A = Q_1 \cdots Q_s \Rightarrow s = 0$ ✓

$r \geq 1$, $r-1 \to r$: Renumbering the P_i's wlog P_1 is minimal in $\{P_1, \dots, P_r\}$.

$P_1 \supseteq Q_1 \cdots Q_s \Rightarrow \exists i: P_1 \supseteq Q_i$. Renumbering, wlog, $P_1 \supseteq Q_1$.

$Q_1 \supseteq P_1 \cdots P_r \Rightarrow \exists j: \quad P_1 \supseteq Q_1 \supseteq P_j \overset{\text{minimality}}{\Longrightarrow} P_j = P_1 \Rightarrow P_1 = Q_1$.

Now, multiplying by P_1^{-1} and using $P_1^{-1} P_1 = A$,

$P_2 \cdots P_r = Q_2 \cdots Q_r$, and the claim follows by induction. $\square$

Thm 8.8 For A a domain, TFAE:

(a) A is a Dedekind domain

(b) Every ideal is a product of prime ideals

(c) Every nonzero ideal is invertible.

Proof: (a) $\Rightarrow$ (b) P8.5 (+ $\underline{0}$ is prime)

(b) $\Rightarrow$ (c) Suffices: every $0 \neq P \in \mathrm{Spec}(A)$ is invertible.

Claim: Every invertible prime ideal is maximal.

Proof of C.: Suppose $P \in \mathrm{Spec}(A)$ is invertible, $P \notin \mathrm{Max}(A)$.

$\Rightarrow \exists a \in A: \quad P + aA \subsetneq A$.

Let $P + aA = P_1 \cdots P_r$, $\quad P + a^2 A = Q_1 \cdots Q_s$, $\quad r, s \geq 1$, $P_i, Q_i \in \mathrm{Spec}(A)$

let $P+aA = P_1 \cdots P_r$, $\quad P+a^2A = Q_1 \cdots Q_s$, $\quad r,s \geq 1$, $P_i, Q_i \in \operatorname{Spec}(A)$ (using (b))

Consider $\pi: A \to A/P =: \bar{A}$, so $\operatorname{Spec}(\bar{A}) \overset{\text{bij}}{\underset{(*)}{\longleftrightarrow}} \{Q \in \operatorname{Spec}(A) : Q \supseteq P\}$

$\Rightarrow \pi(P+aA) = \pi(a)\bar{A} = \pi(P_1) \cdots \pi(P_r)$,

$\pi(P+a^2A) = \pi(a)^2\bar{A} = \pi(Q_1) \cdots \pi(Q_s) = \pi(P_1)^2 \cdots \pi(P_r)^2$

Observe that each $\pi(P_i)$, $\pi(Q_j)$ is invertible

[e.g. $\pi(P_1)\left(\pi(P_2) \cdots \pi(P_r)\pi(a)^{-1}\right) = \bar{A}$]

$\overset{L8.7}{\Longrightarrow}$ $s = 2r$, $(\pi(Q_1), \dots, \pi(Q_s)) = (\pi(P_1), \pi(P_1), \pi(P_2), \pi(P_2), \dots, \pi(P_r), \pi(P_r))$

after renumbering.

$\overset{(*)}{\Longrightarrow}$ $(Q_1, \dots, Q_s) = (P_1, P_1, P_2, P_2, \dots, P_r, P_r)$

$\Rightarrow P + a^2A = (P+aA)^2 \subseteq P^2 + aA$, so $\underline{P \subseteq P^2 + aA}$

$\Rightarrow \underline{P \overset{(**)}{\subseteq} P^2 + aP} = P(P+aA)$ $\quad$ [$\overset{(**)}{\underline{\text{indeed}}}$: let $b \in P \Rightarrow b = c + ad$ with

$c \in P^2$, $d \in A \Rightarrow ad = b - c \in P$, $a \notin P \Rightarrow d \in P$, so $b \in P^2 + aP$.]

$\overset{P \text{ invertible}}{\Longrightarrow}$ $A \subseteq P + aA$ $\lightning$ $\qquad\qquad \square$ (Claim)

Now let $0 \neq P \in \operatorname{Spec}(A)$. Show: P invertible. Let $0 \neq a \in P$

$\Rightarrow aA = P_1 \cdots P_r$ with $r \geq 1$, $P_i \in \operatorname{Spec}(A)$ by (b)

$\Rightarrow \exists i: P_i \subseteq P$, say $P_1 \subseteq P$.

P_1 invertible ($P_1(P_2 \cdots P_r a^{-1}) = A$), so $P_1 \in \operatorname{Max}(A)$ by the Claim

$\Rightarrow P_1 = P$, so P is invertible.

(c) $\Rightarrow$ (a) Invertible ideals are finitely generated (L8.6(3)), so

A is noetherian, $\dim A \leq 1$:

let $0 \neq P \in \operatorname{Spec}(A)$. Show: $P \in \operatorname{Max}(A)$.

Suppose $\underline{P \notin \operatorname{Max}(A)} \to \exists M \in \operatorname{Max}(A): P \subsetneq M$

Observe: $PM^{-1} \subseteq PP^{-1} \subseteq A$, so $PM^{-1} \trianglelefteq A$.

$P = (PM^{-1}) \cdot M \quad \Longrightarrow \quad PM^{-1} \subseteq P \quad \to \quad M^{-1} = P^{-1}PM^{-1} \subseteq P^{-1}P \subseteq A$

Observe: $PM \subseteq PP \subseteq A$, so $PM \trianglelefteq A$.

$$P = (PM^{-1}) \cdot M \underset{\substack{M \not\subseteq P \\ P \text{ prime}}}{\Longrightarrow} PM^{-1} \subseteq P \Rightarrow M^{-1} = P^{-1}PM^{-1} \subseteq P^{-1}P \subseteq A$$

$$\Rightarrow M^{-1}M \subseteq M \subsetneq A \ \lightning$$

<u>A is integrally closed</u> Let $x \in K$ be integral $/A \overset{P6.1}{\Longrightarrow} A[x]$ is a f.g. A-module, hence a frac. ideal $\overset{c)}{\Longrightarrow} A[x]$ invertible $\Rightarrow A[x]A[x]^{-1} = A$

Now: $A[x] = A[x] \cdot A = \underbrace{A[x]A[x]}_{= A[x] \text{ since } A[x] \text{ is a ring}} A[x]^{-1} = A[x]A[x]^{-1} = A$

$\Rightarrow x \in A.$ $\square$

So in a Dedekind domain, the invertible frac. ideals form an abelian group!